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Two outfalls in an estuary: Optimal wasteload allocation

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Abstract. When two outfalls are discharging wastewater into a narrow (rapidly mixed) estuary within a tidal excursion of each other, the pollutant concentrations experienced at the two outfall sites are strongly inter-dependent. It is shown how a given total tidally integrated effluent load can be allocated optimally between the two outfalls so that the peak concentration (in time and position) of the principal contaminant species is minimized. Graphical results show the dependence of the wasteload allocation and of the peak concentration upon the pollutant decay rate, the separation between the outfalls and the fresh water flow along the estuary. Optimization with respect to any one of a mixture of pollutants is close to optimal for a wide range of other pollutants.

Key words: estuary, wastewater, pollutants, mathematical model.

1. Introduction

Desalination of seawater, thermal power generation or waste purification intrinsically involve discharges into the environment of brine, low-temperature waste heat and low-reactivity material. The needs for seawater and for access make it common for the siting to be adjacent to an estuary and for wastewater to be discharged into that estuary. As needs increase, there can be more than one discharge of the same pollutant along a single estuary.

If severe pollution surges are to be avoided, particularly at slack water [1], then the rate of wastewater discharge into a narrow (rapidly mixed) estuary should be matched to the oscillatory dilution capacity. If the water arriving at a single outfall location in a narrow estuary always had zero concentration of the principal contaminant species, then the instantaneous dilution capacity of that arriving water would be proportional to the oscillatory volumetric flow rate. Accordingly, Webb and Tomlinson [2] advocated that discharging be restricted to nonreturning water early on the out-going ebb tide. Holding tanks and pumps would be need to be available, allowing temporary excesses of wastewater to be stored and for the release rate to be controlled. Smith [3] showed that if there is returning pollutant, then the reduced dilution capacity merely involves multiplying the volumetric flow rate by a factor quantifying the incomplete decay. For optimal discharging proportional to the instantaneous dilution capacity, there is no time at which the maximum concentration along a narrow estuary is worse than at any other time: the longitudinal maximum remains constant throughout the tide. Giles [4] gave numerical comparisons between the near optimal (proportional to the volumetric flow rate) and fully optimal discharge strategies.

Bikangaga and Nassehi [5] considered the further complication of there being two wastewater outfalls along a narrow estuary. They showed that the concentration surges are reduced by as much as a factor of 3 when both rates of discharge are proportional to the respective volumetric flow rates. The strategy of proportionality to the volumetric flow rate, takes no account for the modification to the oscillatory dilution capacity from returning pollutant or from pollutant carried from one outfall to the other. The purpose of the present paper is to determine an optimal wasteload allocation and discharging rate at two outfalls which minimizes the peak concentration (in time and in position along the estuary). The optimization gives equal concentration maxima along the estuary immediately downstream of each of the two outfalls. Moreover, those maxima remain constant throughout the tide. An illustrative example reveals that making both discharge rates proportional to the estuarial volume flow rate gives peak concentration levels between 39% and 76% larger than the optimum. Thus, although proportionality to the volumetric flow rate is a good first approximation at removing severe concentration surges [5], optimal wasteload allocation and discharging is a significant further improvement. Management and implementation issues such as control of discharge rates, reliability and cost effectiveness [6] are not addressed.

2. Simplified equations

The conventional equation used to model concentration distributions along narrow estuaries (less than 200 m wide) is the advection dispersion equation with decay [3–4]:

$$A(\partial_t c + u\partial_x c + kc) - \partial_x(AD\partial_x c) = 0. \tag{2.1}$$

Here t is time, x the seaward distance, $A(x,t)$ the cross-sectional area, $u(x,t)$ the cross-sectionally averaged velocity, $c(x,t)$ the cross-sectionally averaged concentration of the most important pollutant (in dimensionless units of mass per unit mass of water), $k(t)$ the decay rate and $D(x,t)$ the longitudinal dispersion coefficient. In the illustrative examples the area $A(x,t)$ and the decay rate $k(t)$ will be constants. The significance of the narrowness of the estuaries is that, soon after and near to the outfall, the cross-sectionally averaged concentration is an appropriate measure of the pollution. For optimal discharging the concentration exiting from both outfalls will be equal and steady. Thus, the concentration distribution along the estuary will be exceptionally flat and the dispersion terms will be of greatly reduced importance [4]. The simplified equations with only advection and decay is

$$\partial_t c + u\partial_x c + kc = 0. \tag{2.2}$$

It deserves emphasis that it is the exceptional flatness of the concentrations that allows the neglect of longitudinal dispersion.

On discharge from the holding tanks at the two outfall locations $x = a_1, a_2$, the contaminant species of interest are assumed to have concentrations $\gamma_1(t)$, $\gamma_2(t)$ which are known and volumetric discharge rates $q_1(t)$, $q_2(t)$ which are to be determined. In the illustrative examples $\gamma_1(t)$, $\gamma(t)$ are equal and constant. The water in the estuary arrives at an outfall with concentration c_{ENTRY} and leaves with increased cross-sectionally averaged concentration c_{EXIT}. A mass balance for the contaminant links the concentration jumps to the respective discharge rates (density changes are neglected)

$$[A_1|u_1| + q_1(t)]c(a_1,t)_{\mathrm{EXIT}} = A_1|u_1|c(a_1,t)_{\mathrm{ENTRY}} + q_1(t)\gamma_1(t) \quad \text{at } x = a_1, \tag{2.3a}$$

$$[A_2|u_2| + q_2(t)]c(a_2,t)_{\mathrm{EXIT}} = A_2|u_2|c(a_2,t)_{\mathrm{ENTRY}} + q_2(t)\gamma_2(t) \quad \text{at } x = a_2. \tag{2.3b}$$

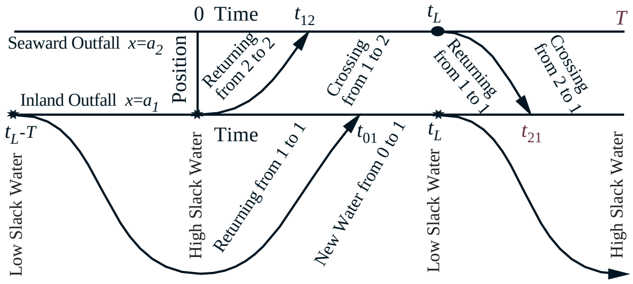

Figure 1. Time-space tracks of water in an estuary with two outfall sites $x = a_1$ and $x = a_2$. Either side of transitional times t_{ij} the water has different pollution histories.

Here $A_1(t)$ is the cross-sectional area and $u_1(t)$ the cross-sectionally averaged velocity on arrival at the inland outfall. The orientation from entry to exit is from inland to seaward on the outgoing ebb tide and reversed on the incoming flood tide. At slack water the arrival flow speed drops to zero and there is no dilution of any discharge made at that time, giving rise to severe pollution surges [1]. These worst excesses can be eliminated by the simple expedient of making the discharge rates $q_j(t)$ proportional to the corresponding incoming estuarial volume flow rates $A_j(t)u_j(t)$.

If the volumetric discharge rates $q_1(t)$, $q_2(t)$ are small relative to the corresponding estuarial volume flow rates (most critically at slack water), then Equations (2.3a, b) can be simplified by the neglect of the left-hand-side $q_j(t)$ terms

$$c(a_1,t)_{\mathrm{EXIT}} = c(a_1,t)_{\mathrm{ENTRY}} + \left(\frac{q_1(t)}{A_1|a_1|}\right)\gamma_1(t) \quad \text{at } x = a_1, \tag{2.4a}$$

$$c(a_2,t)_{\mathrm{EXIT}} = c(a_2,t)_{\mathrm{ENTRY}} + \left(\frac{q_2(t)}{A_2|a_2|}\right)\gamma_2(t) \quad \text{at } x = a_1. \tag{2.4b}$$

Mathematically this linearisation is not needed. However, it does render the subsequent formulae shorter and easier to interpret.

We shall take $t = 0$ to be high slack water, t_L to be low slack water and T the tidal period (the next high slack water). Where the water has been is important as regards the entry concentrations. Although the journeys of the water extend for several periods, periodicity enables us to restrict our attention to the discharge rates in a single representative tidal period.

Figure 1 gives a definition sketch for (lower case) transition times, with the implicit assumption of positive river flow into the estuary. The double subscripts t_{ij} indicate the beginning i and end j positions, with zero signifying far upstream, 1 the inland outfall and 2 the seaward outfall. For example, at the inland outfall $x = a_1$ on the out-going ebb tide prior to the time t_{01}, the water arriving had previously been discharged into on the previous in-coming flood tide at that same outfall. After that transition time t_{01} but before low slack water t_L, the water arriving at the inland outfall t_{01} has never previously been at either of the outfalls (*i.e.* t_{01} indicates beginning 0 and end 1). On the in-coming flood tide prior to the time t_{21}, the water arriving at the inland outfall $x = a_1$ had previously been discharged into on the previous out-going ebb tide at the same outfall. After that transition time t_{21} but before high slack water T, the water arriving at the inland outfall $x = a_1$ had been discharged into on the same flood tide at the seaward outfall (*i.e.* t_{21} indicates beginning 2 and end 1).

For hot arid climates with evaporation volume flux exceeding any river inflow (inverse estuaries [7]), the roles of inland and seaward outfalls would need to be reversed.

3. Exit concentrations at the outfall

As the water moves away from the outfall the pollutant may decay and its concentration will gradually decrease. Thus, the maximum concentration along the estuary occurs at exit from one or both of the outfalls. To investigate the peak concentration (in time and position) it suffices that we solve for the exit concentrations. For compactness, we shall use the notation $c_1(t)$ and $c_2(t)$ for exit concentrations at the two outfalls. If the water arriving at time t has been carried by the tidal flow from the i outfall or has returned to the same $i = j$ outfall, then the 'entry' concentration at time t will be written as a decayed exit concentration from that (upper case) previous time $T_{ij}(t)$. The corresponding exponential decay factor is denoted $E_{ij}(t)$. As in the (lower case) transition times, the double subscripts $_{ij}$ indicate starting i and finishing j sites.

At the inland outfall a_1 in early ebb $0 < t < t_{01}$ or in early flood $t_L < t < t_{21}$, the water is returning to the same outfall (see Figure 1). When we account for the intervening decay since that previous time $T_{11}(t)$, Equation (2.4a) takes the form

$$c_1(t) = E_{11}(t)c_1(T_{11}(t)) + \left(\frac{q_1(t)}{A_1|u_1|}\right)\gamma_1(t) \quad \text{for } 0 < t < t_{01} \text{ or } t_L < t < t_{21}, \tag{3.1a}$$

$$E_{11}(t) = \exp\left(-\int_{T_{11}(t)}^{t} k(t')\,\mathrm{d}t'\right) \quad \text{(from site 1 to site 1)}. \tag{3.1b}$$

At the inland outfall a_1 in late ebb $t_{01} < t < t_L$, net river flow into the estuary can give rise to a time span in which new water arrives which has never previously passed either outfall (see Figure 1) and therefore has zero incoming concentration. The exiting concentration is

$$c_1(t)\left(\frac{q_1(t)}{A_1|u_1|}\right)\gamma_1(t) \quad \text{for } t_{01} < t < t_L \text{ (new water)}. \tag{3.2}$$

At the inland outfall a_1 in late flood $t_{21} < t < T$, water can be arriving which departed from the seaward a_2 outfall at the earlier time $T_{21}(t)$. The decay plus the new discharge give the late flood exiting concentration

$$c_1(t) = E_{21}(t)c_2(T_{21}(t)) + \left(\frac{q_1(t)}{A_1|u_1|}\right)\gamma_1(t) \quad \text{for } t_{21} < t < T, \tag{3.3a}$$

$$E_{21}(t) = \exp\left(-\int_{T_{21}(t)}^{t} k(t')\,\mathrm{d}t'\right) \quad \text{(from site 2 to site 1)}. \tag{3.3b}$$

At the seaward outfall a_2 the demarcation of time regimes is different (see Figure 1). On early ebb $0 < t < t_{12}$ and throughout the flood $t_L < t < T$, we denote the previous time water left the same outfall as $T_{22}(t)$. The exiting concentration is

$$c_2(t) = E_{22}(t)c_2(T_{22}(t)) + \left(\frac{q_2(t)}{A_2|u_2|}\right)\gamma_2(t) \quad \text{for } 0 < t < t_{12} \text{ or } t_L < t < T, \tag{3.4a}$$

$$E_{22}(t) = \exp\left(-\int_{T_{22}(t)}^{t} k(t')\,\mathrm{d}t'\right) \quad \text{(from site 2 to site 2).} \tag{3.4b}$$

At the seaward outfall a_2 on late ebb $t_{12} < t < t_L$, river flow can give rise to a time span in which the water has come from the a_1 outfall at time $T_{12}(t)$

$$c_2(t) = E_{12}(t)c_1(T_{12}(t)) + \left(\frac{q_2(t)}{A_2|u_2|}\right)\gamma_2(t) \quad \text{for } t_{12} < t < t_L, \tag{3.5a}$$

$$E_{12}(t) = \exp\left(-\int_{T_{12}(t)}^{t} k(t')\,\mathrm{d}t'\right) \quad \text{(from site 1 to site 2).} \tag{3.5b}$$

4. Optimal sharing of the pollution load

Reducing the peak exit concentration $c_i(t)$ at one of the outfalls by reducing the discharge rate $q_i(t)$ at that outfall, has to be compensated by increasing one or both of the discharge rates $q_j(t')$ at some nonpeak times (to keep the total amount discharged into the estuary over a tidal cycle fixed). Hence the nonpeak exit concentrations $c_j(t')$ are increased. The optimization giving the lowest possible peak concentration would be achieved when, at all times and at both outfalls the exit concentrations have a constant value C_{OPT}.

With $c_1 = c_2 = C_{\mathrm{OPT}}$ Equations (3.1–3.5) can be re-arranged to give the optimal discharge rates

$$q_1(t)\gamma_1(t) = C_{\mathrm{OPT}}A_1|u_1|[1 - E_{11}(t)] \quad 0 < t < t_{01}, \tag{4.1a}$$

$$q_1(t)\gamma_1(t) = C_{\mathrm{OPT}}A_1|u_1| \quad t_{01} < t < t_L, \tag{4.1b}$$

$$q_1(t)\gamma_1(t) = C_{\mathrm{OPT}}A_1|u_1|[1 - E_{11}(t)] \quad t_L < t < t_{21}, \tag{4.1c}$$

$$q_1(t)\gamma_1(t) = C_{\mathrm{OPT}}A_1|u_1|[1 - E_{21}(t)] \quad t_{21} < t < T, \tag{4.1d}$$

$$q_2(t)\gamma_2(t) = C_{\mathrm{OPT}}A_2|u_2|[1 - E_{22}(t)] \quad 0 < t < t_{12}, \tag{4.1e}$$

$$q_2(t)\gamma_2(t) = C_{\mathrm{OPT}}A_2|u_2|[1 - E_{12}(t)] \quad t_{12} < t < t_L, \tag{4.1f}$$

$$q_2(t)\gamma_2(t) = C_{\mathrm{OPT}}A_2|u_2|[1 - E_{22}(t)] \quad t_L < t < T. \tag{4.1g}$$

For nondecaying effluent (such as brine) there is no decay, $E_{ij} = 1$ and discharging is confined to the inland outfall during the new water time span (4.1b). In the opposite limit of an infinitely fast decay (such as a rapidly consumed nutrient), the decay factors E_{ij} tend to zero and the optimal discharge rates $q_j(t)$ are proportional to the volume flow rates $A_j|u_j|$.

Consistent with the use of double subscripts to characterize the different water histories (3.1–3.5), we can split the tidal averages for the wasteload allocations at the two outfalls into distinct contributions

$$\langle q_1\gamma_1\rangle = \langle q_1\gamma_1\rangle_{11} + \langle q_1\gamma_1\rangle_{01} + \langle q_1\gamma_1\rangle_{21}, \tag{4.2a}$$

$$\langle q_2\gamma_2\rangle = \langle q_2\gamma_2\rangle_{22} + \langle q_2\gamma_2\rangle_{12}. \tag{4.2b}$$

Here angle brackets denote tidal averages. For optimal sharing we can evaluate the maximum concentration C_{OPT} in terms of any one or linear combination of $\langle q_1\gamma_1\rangle$ and $\langle q_2\gamma_2\rangle$

$$\langle q_1\gamma_1\rangle = C_{\mathrm{OPT}}\{\langle A_1|u_1|[1-E_{11}]\rangle_{11} + \langle A_1|u_1|\rangle_{01} + \langle A_1|u_1|[1-E_{21}]\rangle_{21}\}, \tag{4.3a}$$

$$\langle q_2\gamma_2\rangle = C_{\mathrm{OPT}}\{\langle A_2|u_2|[1-E_{22}]\rangle_{22} + \langle A_2|u_2|[1-E_{12}]\rangle_{12}\}. \tag{4.3b}$$

Conversely, if there are environmental standards which impose a fixed upper bound for C_{OPT}, then Equations (4.3a,b) determine the maximum permissible tidal averaged discharge rates $\langle q_1\gamma_1\rangle$ and $\langle q_2\gamma_2\rangle$.

If it is possible to choose how a given total tidal wasteload $\langle q_1\gamma_1\rangle + \langle q_2\gamma_2\rangle$ of the most important contaminant species is shared between the two sites, then the ratio $\langle q_1\gamma_1\rangle : \langle q_2\gamma_2\rangle$ follows directly from the ratio of equations (4.3a,b). The new water time range (4.1b) tends to make the amount $\langle q_1\gamma_1\rangle$ discharged at the inland outfall always be greater than the amount $\langle q_2\gamma_2\rangle$ discharged at the seaward outfall.

5. Piecewise sinusoidal current

We use an illustrative example which has flexibility to incorporate river flow, yet retains the convenience of explicit formulae for the transition times t_{ij} and previous discharge times $T_{ij}(t)$. The cross-sectional area A is modelled as being constant and the tidal current as being piecewise sinusoidal

$$u = U(1+F)\sin\left(\frac{2\pi}{1+F}\left(\frac{t}{T}\right)\right) \text{ ebb } 0 < \frac{t}{T} < \frac{t_L}{T} = \frac{1+F}{2}, \tag{5.1a}$$

$$u = -U(1-F)\sin\left(\frac{2\pi}{1-F}\left(1-\frac{t}{T}\right)\right) \text{ flood } \frac{1+F}{2} = \frac{t_L}{T} < \frac{t}{T} < 1, \tag{5.1b}$$

$$F = \frac{\pi\langle u\rangle}{4U}. \tag{5.1c}$$

Here U is an effective amplitude for the tidal velocity, F is a dimensionless characterization for the relative strength of the mean river flow $\langle u\rangle$ and T is the tidal period. The mean flow slightly speeds up and prolongs the ebb, with slowing and shortening of the flood (by amounts proportional to F). A natural excursion length scale to associate with the tidal current is

$$L = UT/\pi. \tag{5.2}$$

On the flood the water can have returned from the seaward outfall site a_2 only if

$$\frac{a_2 - a_1}{L} < (1-F)^2. \tag{5.3}$$

In the limit of rapidly decaying pollutant, with nonzero separation, the optimized discharges (4.3a,b) give a peak concentration

$$c_\infty = \frac{\pi(\langle q_1\gamma_1\rangle + \langle q_2\gamma_2\rangle)}{4(1+F^2)AU}. \tag{5.4a}$$

In the further limit of negligible river flow $F = 0$, this provides the (lower bound) reference concentration C_{REF} used later for normalization in Figures 5, 6

$$C_{\mathrm{REF}} = \frac{\pi(\langle q_1\gamma_1\rangle + \langle q_2\gamma_2\rangle)}{4AU}. \tag{5.4b}$$

For the inland outfall the transition times, as indicated in Figure 1, are

$$\frac{t_{01}}{T} = \frac{1+F}{\pi}\arcsin\left\{\frac{1-F}{1+F}\right\},$$

$$\frac{t_{21}}{T} = 1 - \frac{(1-F)}{\pi}\arccos\left\{\left[\frac{a_2-a_1}{L}\right]^{\frac{1}{2}}\frac{1}{1-F}\right\}. \tag{5.5a,b}$$

On the early ebb the water is returning from a time just prior to high water slack

$$\frac{T_{11}(t)}{T} = -\frac{(1-F)}{\pi}\arcsin\left\{\frac{1+F}{1-F}\sin\left(\frac{\pi}{1+F}\left(\frac{t}{T}\right)\right)\right\} \quad \text{for } 0 < t < t_{01}. \tag{5.6a}$$

In the new water time range $t_{01} < t < t_L$ there is no previous time at the inland outfall. On early flood the water is returning from just prior to low water slack

$$\frac{T_{11}(t)}{T} = \frac{(1+F)}{\pi}\arccos\left\{\frac{1-F}{1+F}\cos\left(\frac{\pi}{1-F}\left(1-\frac{t}{T}\right)\right)\right\} \quad \text{for } t_L < t < t_{21}. \tag{5.6b}$$

On the late flood the water has travelled from the seaward outfall, which it left at time

$$\begin{aligned}\frac{T_{21}(t)}{T} \;=\; 1 - \frac{(1-F)}{\pi}\arccos\Bigg\{\Bigg[&\cos^2\left(\frac{\pi}{1-F}\left(1-\frac{t}{T}\right)\right)\\ &- \left[\frac{a_2-a_1}{L}\right]\frac{1}{(1-F)^2}\Bigg]^{\frac{1}{2}}\Bigg\} \quad \text{for } t_{21} < t < T.\end{aligned} \tag{5.6c}$$

For the seaward outfall the transition time, as indicated in Figure 1, is

$$\frac{t_{12}}{T} = \frac{1+F}{\pi}\arcsin\left\{\left[\frac{a_1-a_2}{L}\right]^{\frac{1}{2}}\frac{1}{(1+F)}\right\}. \tag{5.7}$$

On the early ebb the water is returning from just before high slack water

$$\frac{T_{22}(t)}{T} = -\frac{(1-F)}{\pi}\arcsin\left\{\frac{1+F}{1-F}\sin\left(\frac{\pi}{1+F}\left(\frac{t}{T}\right)\right)\right\} \quad \text{for } 0 < t < t_{12}, \tag{5.8a}$$

On later ebb the water at the seaward outfall was previously at the inland outfall

$$\begin{aligned}\frac{T_{12}(t)}{T} \;=\; \frac{(1+F)}{\pi}\arcsin\Bigg\{\Bigg[&\sin^2\left(\frac{\pi}{1+F}\left(\frac{t}{T}\right)\right)\\ &- \left(\frac{a_2-a_1}{L}\right)\frac{1}{(1+F)^2}\Bigg]^{\frac{1}{2}}\Bigg\} \quad \text{for } t_{12} < t < t_L.\end{aligned} \tag{5.8b}$$

Throughout the flood the water is returning from the previous ebb

$$\frac{T_{22}(t)}{T} = \frac{(1+F)}{\pi} \arccos\left\{\frac{1-F}{1+F}\cos\left(\frac{\pi}{1-F}\left(t-\frac{t}{T}\right)\right)\right\} \quad \text{for } t_L < t < T. \tag{5.8c}$$

Since the flow does not vary along the estuary, the return functions $T_{11}(t)$, $T_{22}(t)$ only differ from each other in their ranges of application.

6. Illustrative examples

There are three dimensionless parameters which characterize the geometry, pollutant and flow: the separation $(a_2 - a_1)/L$ between the discharges, the river to tidal flow parameter F and the decay kT of the most important pollutant. As a quantitative illustration we consider an estuary with a semi-diurnal tide of velocity amplitude $0{\cdot}7\,\mathrm{ms}^{-1}$, giving an natural excursion length $L = 10$ km. The outfalls are 5 km apart, the tidally averaged mean river flow velocity is $0{\cdot}09\,\mathrm{ms}^{-1}$ and most important pollutant has a constant e-folding decay time of a day (2 tides). In round figures, the dimensionless characterization of this illustrative case is:

$$\frac{a_2 - a_1}{L} = 0{\cdot}5, \quad F = 0{\cdot}1, \quad kT = 0{\cdot}5. \tag{6.1}$$

Figure 2 illustrates the time-dependence of $q_1(t)$ and $10q_2(t)$ as fractions of the tidally averaged total wasteload $\langle q_1\rangle + \langle q_2\rangle$. As in all the illustrative examples, the concentrations prior to discharge $\gamma_1(t)$, $\gamma_2(t)$ are assumed to be equal and constant. The optimal sharing is strongly in favour of the inland discharge

$$\langle q_1\rangle : \langle q_2\rangle = 0{\cdot}699 : 0{\cdot}301. \tag{6.2}$$

At the seaward outfall (amplified in Figure 2 by a factor of 10) there is an abrupt reduction in discharge rate at time $t_{12}/T = 0{\cdot}244$ during ebb, when recently polluted water arrives from the inland outfall. At the inland outfall there is a sudden surge in the discharge rate at time $t_{01}/T = 0{\cdot}336$ during ebb, when 'new' zero-concentration water arrives. At the inland outfall there is an abrupt reduction in discharge rate at time $t_{21}/T = 0{\cdot}801$ during flood, when recently polluted water arrives from the seaward outfall. Apart from the jumps, the discharge rates resemble the flow speed. In particular, there is zero discharge at high and low slack waters.

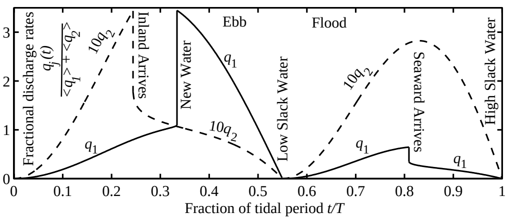

Figure 2. The optimal discharge rates $q_1(t)$, $q_2(t)$ at two outfalls as fractions of their tidally averaged sum $\langle q_1\rangle + \langle q_2\rangle$. There are abrupt changes in discharge rates at transition times, in particular, when unpolleted 'new water' arrives at the inland outfall.

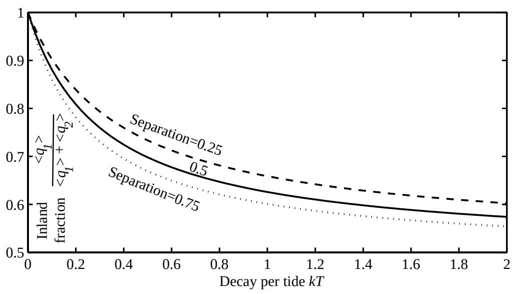

Figure 3. The influence of decay rate and separation between the outfalls (relative to tidal excursion), upon the fraction of the total pollution load that should be discharged from the inland outfall in order to minimize the peak concentration. The river to tidal flow parameter is $F = 0{\cdot}1$.

For the optimized discharges the most marked variations are with respect to the decay rate kT (*e.g*. differences between brine, temperature and rapidly consumed nutrients). Accordingly, Figure 3 shows the fraction of the tidally averaged load to be discharged from the inland outfall as a function of kT for three different separations with the river flow fixed

$$\frac{a_2 - a_1}{L} = 0{\cdot}25,\, 0{\cdot}5,\, 0{\cdot}75, \quad F = 0{\cdot}1. \tag{6.3a}$$

For more decay or greater separation between the outfalls, the water arriving at the seaward outfall is less polluted and can receive a greater discharge. Hence, $\langle q_2 \rangle$ is increased and $\langle q_1 \rangle$ decreased. For extremely large decay rates very little pollutant from one outfall reaches the other. Thus, the optimal inland and seaward discharge rates become equal and the inland share of the total tidal average asymptotes to a half.

Figure 4 (with a kT range twice that of Figure 3) shows the fraction of the tidally averaged load to be discharged from the inland outfall with the separation fixed and three different river flow strengths

$$\frac{a_2 - a_1}{L} = 0{\cdot}5,\, F = 0{\cdot}04,\, 0{\cdot}1,\, 0{\cdot}25. \tag{6.3b}$$

Large river flow (increased seawards velocity) reduces the time and amount of decay for polluted water arriving at the seaward outfall. Thus, the inland optimal share of the discharge increases with the river flow.

Figure 5 shows graphs of the peak concentration for the three different separations (scaled relative to the lower bound reference concentration C_{REF}). An additional dot-dash curve has been included in Figure 5 which shows the peak concentration for a discharge proportional to the estuarial volumetric flow when applied to the same outfall separation and nonequally shared tidally averaged wasteloads as the continuous curve in the same figure. The flow-matched curve is 39% above the optimum cooperative strategy for nondecaying pollutant ($kT = 0$) increasing to 76% for the highest decay rate illustrated in Figure 5. As we might expect, it is when the water has returned most recently and most often (immediately after low slack water) that a discharge proportional to the estuarial volumetric flow performs worst.

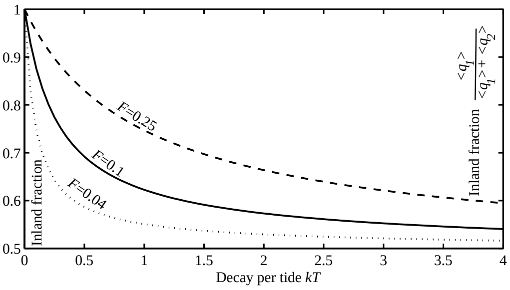

Figure 4. The influence of decay rate kT and river to tidal flow parameter F, upon the fraction of the total pollution load that should be discharged from the inland outfall in order to minimize the worst pollution. The separation between outfalls is 0·5 tidal excursions.

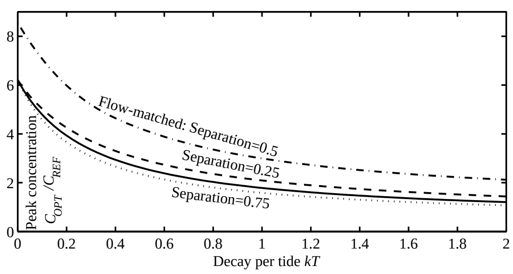

Figure 5. Peak concentration (same at both outfalls) for optimal wasteload allocation as functions of the decay rate kT, for three different fractions of the tidal excursion apart. For 0·5 tidal excursions separation, a discharge proportional to the flow gives the higher dash-dot peak pollution instead the continuous curve.

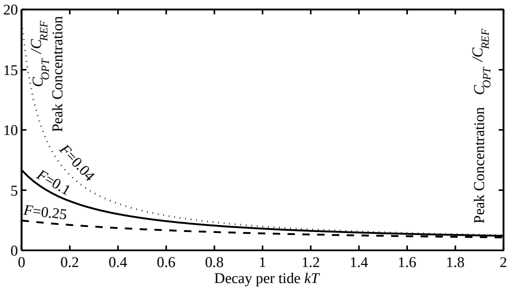

Figure 6. Peak concentration (same at both outfalls) for optimal wasteload allocation as functions of the decay rate kT, for three different values of the river to tidal flow parameter F. The separation between outfalls is 0·5 tidal excursions.

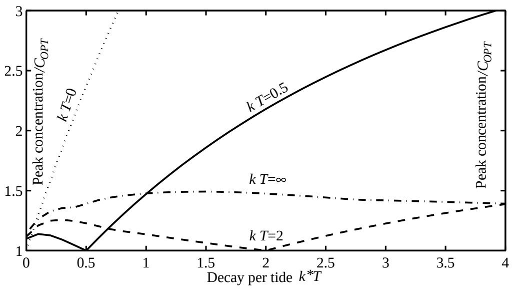

Figure 7. Peak concentration relative to the optimized minimum C_{OPT}, when the decay rate k^* differs from the reference value k that was used in selecting the wasteload allocation to the outfalls. The outfall separation is 0·5 tidal excursions and the river to tidal flow parameter is $F = 0{\cdot}1$.

Figure 6 gives the peak concentration for the three river flow rates (scaled relative to the reference concentration C_{REF}). The kT range is the same as in Figures 3, 5 and half that of Figure 4. For the dotted $F = 0{\cdot}04$ curve near $kT = 0$, the combination of repeated returning in low river flow conditions and of small decay gives high peak concentration levels. The sensitivity of pollution levels to drought decreases markedly as the decay rate of the most important pollutant increases *i.e.* flushing is unimportant if the pollutant decays rapidly.

7. Mixtures of pollutants

The wasteload allocation and the time-dependent rates of discharge are optimized to minimize the peak concentration of the most important constituent (with decay rate k). However, the wastewater from (say) a desalination plant would have a different temperature and nutrient concentration from that of the receiving waters. What is the effect upon the peak concentration of constituents with other decay rates k^*?

Figure 7 show the peak concentration levels (relative to the optimum shown as the continuous curve in Figures 5 and 6) for four different kT values with the separation and river flow fixed

$$\frac{a_2 - a_1}{L} = 0{\cdot}5, \quad kT = 0, 0{\cdot}5, 2, \infty, \quad F = 0{\cdot}1. \tag{7.1}$$

The scaling of the peak concentrations makes a value of unity optimal, as is achieved when $k^* = k$. If the most important pollutant has infinitely fast decay $kT = \infty$, then the wasteload is shared equally between the two outfalls. The relative concentrations, from the ratio of the dash-dot to continuous curves in Figure 5 and from the dash-dot curve in Figure 7, differ because the sharing of the wasteload is different.

If the most important pollutant has zero decay $kT = 0$ (*e.g.* brine), then the low river flow $F = 0{\cdot}1$ implies that there is a brief very large discharge restricted to the inland outfall and to the new water in late ebb tide. The repeated returning without decay precludes additional discharging at other times or at the seaward outfall. For other constituents, with k^*T greater than F, there will have been significant decay before the final return to the inland outfall in early ebb tide. Therefore, there would have been opportunity for significant discharging at other times or at the other outfall. The dotted curve in Figure 7 quantifies how large the peak pollution becomes relative to the achievable minimum as k^*T increases away from zero.

With the exception of $kT = 0$, it is only for constituents with decay rates k^* far from k that the sharing and time-dependence of the discharging deviate substantially above the achievable minimum. As in the single outfall situation [8], we can anticipate that there is only weak dependence upon any complicating features such as day–night variability of heat loss or nutrient consumption rate $k(t)$. This is suggestive that the cooperative pollution minimization is not only effective but robust.

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